Course in Nonlinear FEM

Introduction

Outline

Lecture 1 – Introduction
Lecture 2 – Geometric nonlinearity
Lecture 3 – Material nonlinearity
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Lecture 5 – Geometric nonlinearity revisited
Lecture 6 – Issues in nonlinear FEA
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Lecture 9 – Dynamics
Lecture 10 – Dynamics continued

Nonlinear FEM

Lecture 1 – Introduction, Cook [17.1]:
- Types of nonlinear problems
- Definitions

Lecture 2 – Geometric nonlinearity, Cook [17.10, 18.1-18.6]:
- Linear buckling or eigen buckling
- Prestress and stress stiffening
- Nonlinear buckling and imperfections
- Solution methods

Lecture 3 – Material nonlinearity, Cook [17.3, 17.4]:
- Plasticity systems
- Yield criteria

Lecture 4 – Material nonlinearity revisited, Cook [17.6, 17.2]:
- Flow rules
- Hardening rules
- Tangent stiffness

Nonlinear FEM

Lecture 5 – Geometric nonlinearity revisited, Cook [17.9, 17.3-17.4]:

- The incremental equation of equilibrium
- The nonlinear strain-displacement matrix
- The tangent-stiffness matrix
- Strain measures

Lecture 6 – Issues in nonlinear FEA, Cook [17.2, 17.9-17.10]:

– Solution methods and strategies
– Convergence and stop criteria
– Postprocessing/Results
– Troubleshooting

Nonlinear FEM

Lecture 7 – Contact nonlinearity, Cook [17.8]:
- Contact applications
- Contact kinematics
- Contact algorithms

Lecture 8 – Contact nonlinearity continued, Cook [17.8]:
- Issues in FE contact analysis/troubleshooting

Lecture 9 – Dynamics, Cook [11.1-11.5]:
- Solution methods
- Implicit methods
- Explicit methods

Lecture 10 – Dynamics continued, Cook [11.11-11.18]:
- Dynamic problems and models
- Damping
- Issues in FE dynamic analysis/troubleshooting

References

- **[ANSYS]** ANSYS 10.0 Documentation (installed with ANSYS):
  - Basic Analysis Procedures
  - Advanced Analysis Techniques
  - Modeling and Meshing Guide
  - Structural Analysis Guide
  - Thermal Analysis Guide
  - APDL Programmer's Guide
  - ANSYS Tutorials
- **[Cook]** Cook, R. D.; Concepts and applications of finite element analysis, John Wiley & Sons

Program

- Types of nonlinear structural problems
- What is nonlinearity
  - How is it defined/characterized
  - Where does it occur
  - What are the basic mechanisms
- Why/When to consider nonlinear analysis
- Structures which exhibit a characteristic nonlinear behavior
- Definition of common terms

Overview

Types of nonlinear structural problems

- Material nonlinearity
- Geometric nonlinearity
- Dynamic problems
- Contact problems

What is nonlinearity

In mathematics, **nonlinear** systems represent systems whose behavior is not expressible as a sum of the behaviors of its descriptors. In particular, the behavior of nonlinear systems is not subject to the principle of superposition, as linear systems are. Crudely, a nonlinear system is one whose behavior is not simply the sum of its parts or their multiples.

What is nonlinearity

- Linearity of a system allows investigators to make certain mathematical assumptions and approximations, allowing for easier computation of results. In nonlinear systems these assumptions cannot be made.
- Since nonlinear systems are not equal to the sum of their parts, they are often difficult (or impossible) to model, and their behavior with respect to a given variable (for example, time) is extremely difficult to predict.
- When modeling non-linear systems, therefore, it is common to approximate them as linear, where possible.

What is nonlinearity

- **Nonlinear systems**
  - Nonlinear equations and functions are of interest to physicists and mathematicians because most physical systems are inherently nonlinear in nature. Physical examples of linear systems are relatively rare. Nonlinear equations are difficult to solve and give rise to interesting phenomena such as chaos. A linear equation can be described by using a linear operator, L. A linear equation in some unknown u has the form $Lu = 0$.
  - A nonlinear equation is an equation of the form $F(u) = 0$, for some unknown u.
  - In order to solve any equation, one needs to decide in what mathematical space the solution u is found. It might be that u is a real number, a vector or perhaps a function with some properties.

What is nonlinearity

- **Nonlinear systems**
  - The solutions of linear equations can in general be described as a superposition of other solutions of the same equation. This makes linear equations particularly easy to solve.
  - Nonlinear equations are more complex, and much harder to understand because of their lack of simple superposed solutions.
  - For nonlinear equations the solutions to the equations do not in general form a vector space and cannot (in general) be superposed (added together) to produce new solutions. This makes solving the equations much harder than in linear systems.

What is nonlinearity

- Define linearity:

  A function $f(x_1,x_2,\ldots,x_n)$ is linear IF

  $f(ax) = a\, f(x)$ and $f(x+y) = f(x) + f(y)$
- A nonlinear system breaks either or both of these equalities
- Most nonlinear systems are irreversible
- We must modify our analysis technique to account for nonlinear responses.

What is nonlinearity

Examples of nonlinear equations

- **general relativity**
- **the Navier-Stokes equations of fluid dynamics**
- **systems with solutions as solutions**
- **nonlinear optics**
- **the Earth's weather system**
- **balancing a robot unicycle**
- **Boltzmann transport equation**
- **Korteweg-de Vries equation**
- **sine-Gordon equation**
- **nonlinear Schroedinger equation**
- **chaos theory, fractals**
- **Lyapunov stability and non-linear control systems**
- ***etc.***

Why/When to consider nonlinear analysis

Application	*Explanation*
Strength analysis	How much load can the structure support before global failure occurs?
Deflection analysis	When deflection control is of primary importance
Stability analysis	Finding critical points (limit points or bifurcation points) closest to operational range
Service configuration analysis	Finding the "operational" equilibrium form of certain slender structures when the fabrication and service configurations are quite different (e.g. cables, inflatable structures, helicoids)
Reserve strength analysis	Finding the load carrying capacity beyond critical points to assess safety under abnormal conditions.
Progressive failure analysis	A variant of stability and strength analysis in which progressive deterioration (e.g. cracking) is considered.
Envelope analysis	A combination of previous analyses in which multiple parameters are varied and the strength information thus obtained is condensed into failure envelopes.

Nonlinear behavior

- Structures which exhibit a characteristic nonlinear behavior
  - Hydraulic piston, drill pipe – bar/rod
  - Offshore structure, truss and frame structures - beam/pipe
  - Crank shaft - solid
  - Wing panel, wind-turbine blade, silo – plate/shell

  - Deep drawing
  - Collisions, crash test
  - Push-over analysis

Linear Problem

Linear FEA is based on

- linearized geometrical equations (strain-displacement relations):
  $\{\varepsilon\} = [B]\{d\}$
- linearized constitutive equations (stress-strain relations):
  $\{\sigma\} = [E]\{\varepsilon\} = [E][B]\{d\}$
- equations of equilibrium: $\{R^i\} = \{R^e\}$, linear so that:
  $[K]\{D\} = \{R^e\}$

and suitable boundary conditions, i.e. the assumptions made are often crude.

Linear Problem

$$[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{R}\}$$

$$[\mathbf{K}] \neq [\mathbf{K}(\{\mathbf{D}\})]$$

$$\{\mathbf{R}\} \neq \{\mathbf{R}(\{\mathbf{D}\})\}$$

Stiffness and Forces are not functions of displacements.

Nonlinear Problem

The nonlinear behaviour occur as stiffness and loads become functions of displacement or deformation, i.e. in

$$[K]\{D\} = \{R\}$$

both the structural stiffness matrix $[K]$ and possibly the load vector $\{R\}$ become functions of the displacements $\{D\}$. Therefore it is not possible to solve for $\{D\}$ immediately as $[K]$ and $\{R\}$ is not known in advance.

Therefore an iterative process is needed to obtain $\{D\}$ and the associated $[K]$ and $\{R\}$ such that $[K]\{D\}$ with $\{R\}$.

Nonlinear Problem

$$[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{R}\}$$

$$[\mathbf{K}] = [\mathbf{K}(\{\mathbf{D}\})]$$

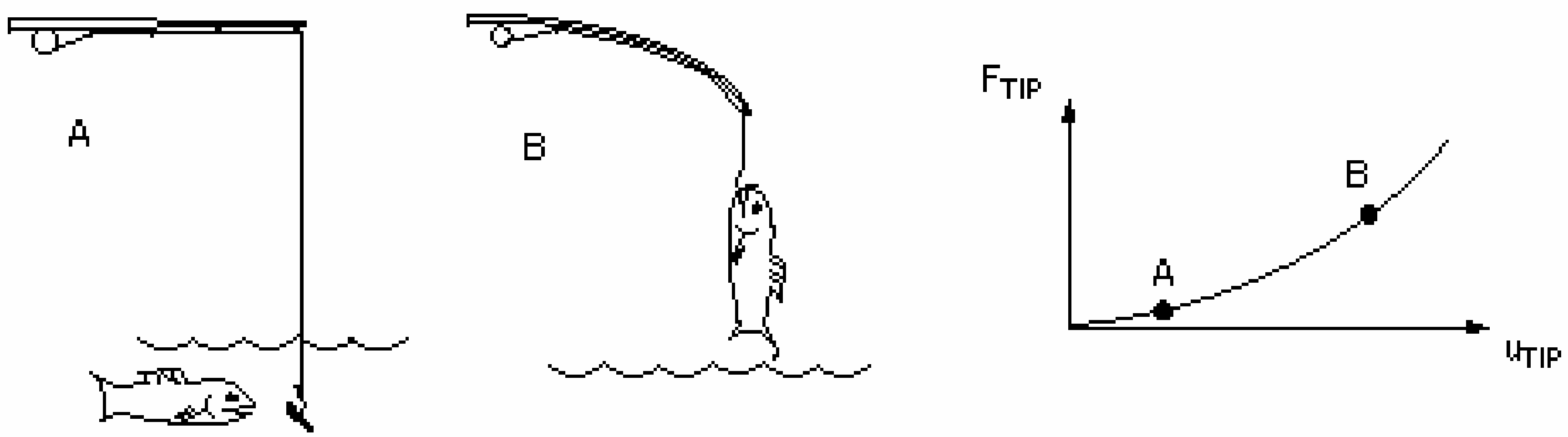
$$\{\mathbf{R}\} = \{\mathbf{R}(\{\mathbf{D}\})\}$$

Stiffness and Forces are functions of displacements.

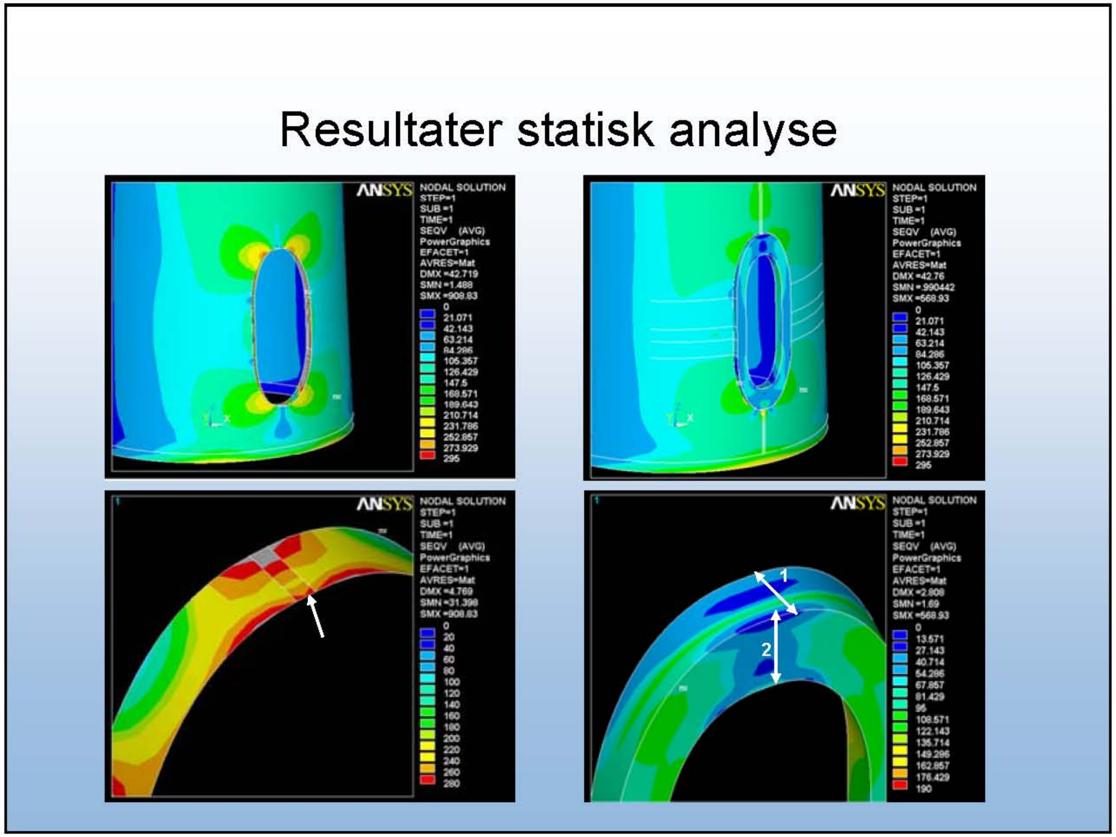
Geometric nonlinearity

- Geometric nonlinearity is characterized by "large" displacements and/or rotations.

Figure 8-2 A fishing rod demonstrates geometric nonlinearity



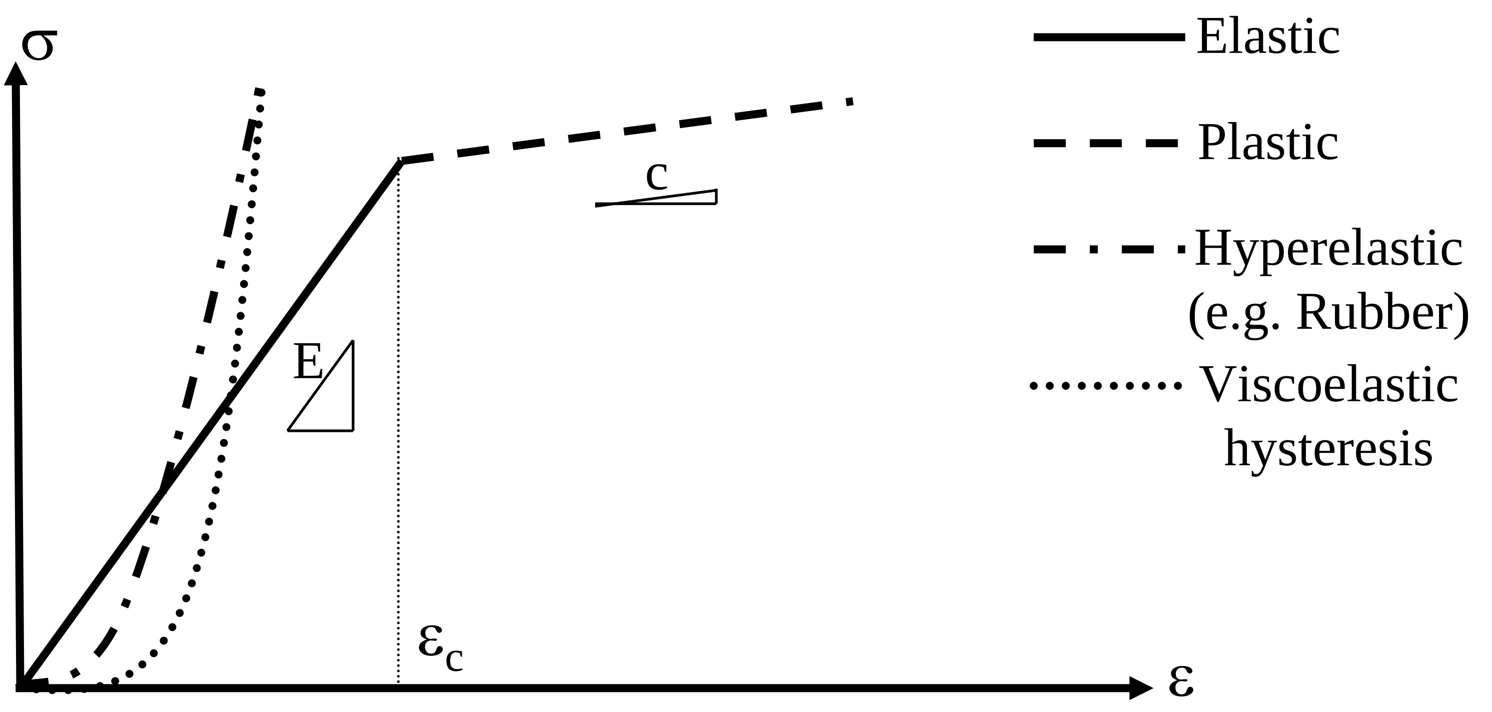
Geometric nonlinearity



13

Material nonlinearity

- Material types



Material nonlinearity

- Material formulae
  - Elastic $\sigma = E\varepsilon$
  - Plastic (e.g. bilinear)

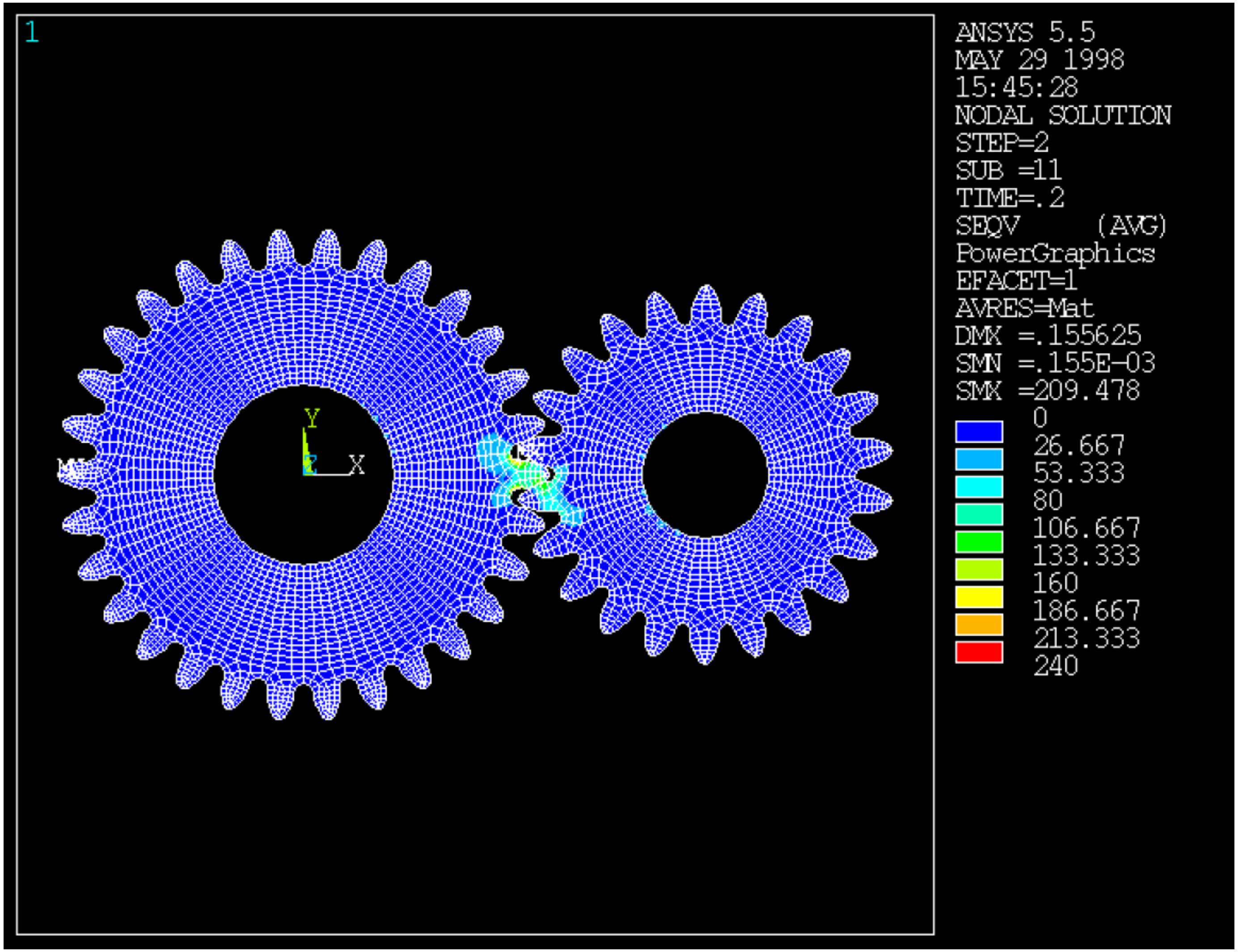
$$\sigma = \begin{cases} E\varepsilon, & \text{for}\, \varepsilon < \varepsilon_c \\ E\varepsilon_c + c(\varepsilon - \varepsilon_c), & \text{for}\, \varepsilon \geq \varepsilon_c \end{cases}$$

  - Hyperelastic (e.g. Mooney Rivlin model)

$$\sigma = 2C_{10}\left(e^{2\varepsilon} - e^{-\varepsilon}\right) + 2C_{01}\left(e^{\varepsilon} - e^{-2\varepsilon}\right)$$

    - Where C_{10} and C_{01} are material parameters

Contact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MAY 29 1998
15:45:28
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SUB =11
TIME=.2
SEQV (AVG)
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.155625
SMN =.155E-03
SMX =209.478
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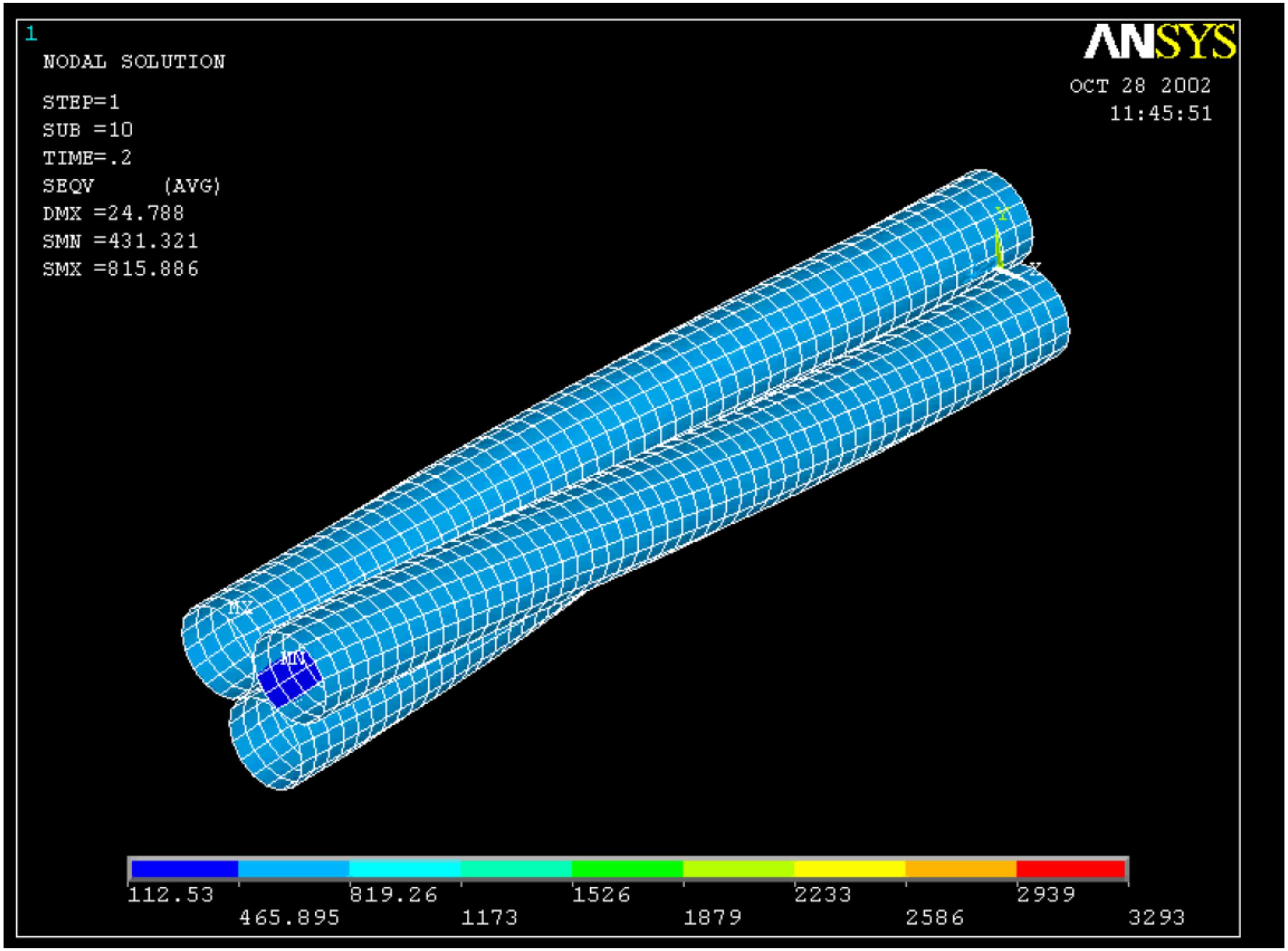
53.333
80
106.667
133.333
160
186.667
213.333
240
Y
Z
X

Contact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ANSYS
OCT 28 2002
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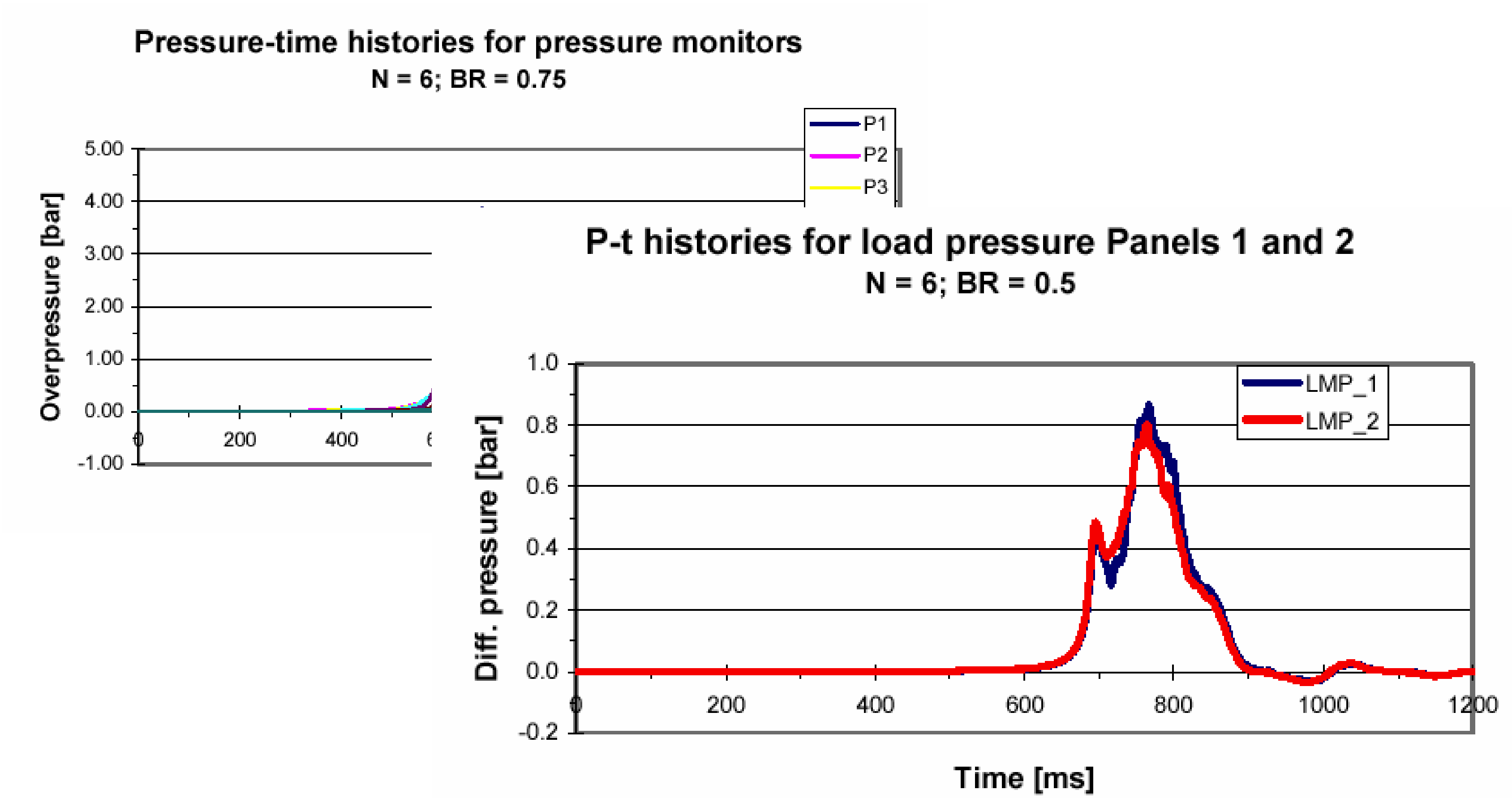
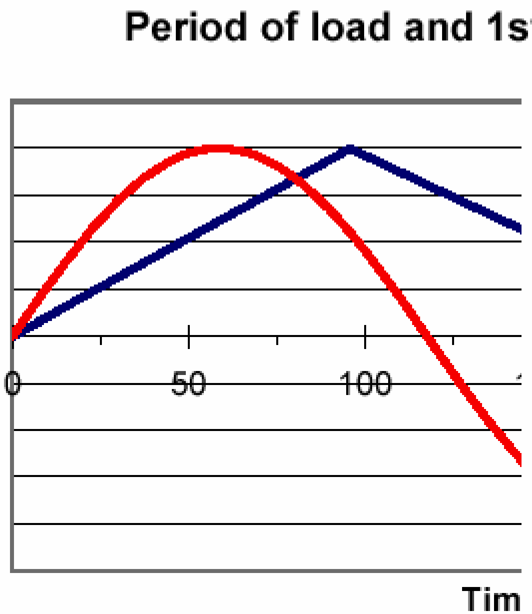
1526
1879
2233
2586
2939
3293

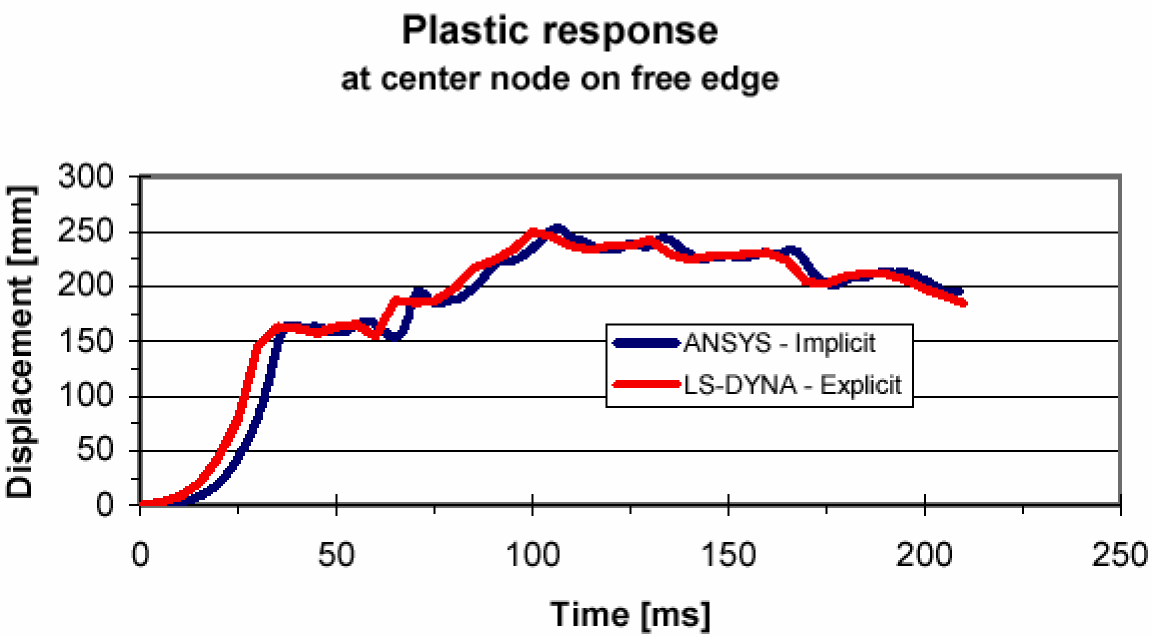
Dynamics


Pressure-time histories for pressure monitors
N = 6; BR = 0.75
P1
P2
P3
Overpressure [bar]
5.00
4.00
3.00
2.00
1.00
0.00
-1.00
0
200
400
P-t histories for load pressure Panels 1 and 2
N = 6; BR = 0.5
LMP_1
LMP_2
Diff. pressure [bar]
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
200
400
600
800
1000
1200
Time [ms]

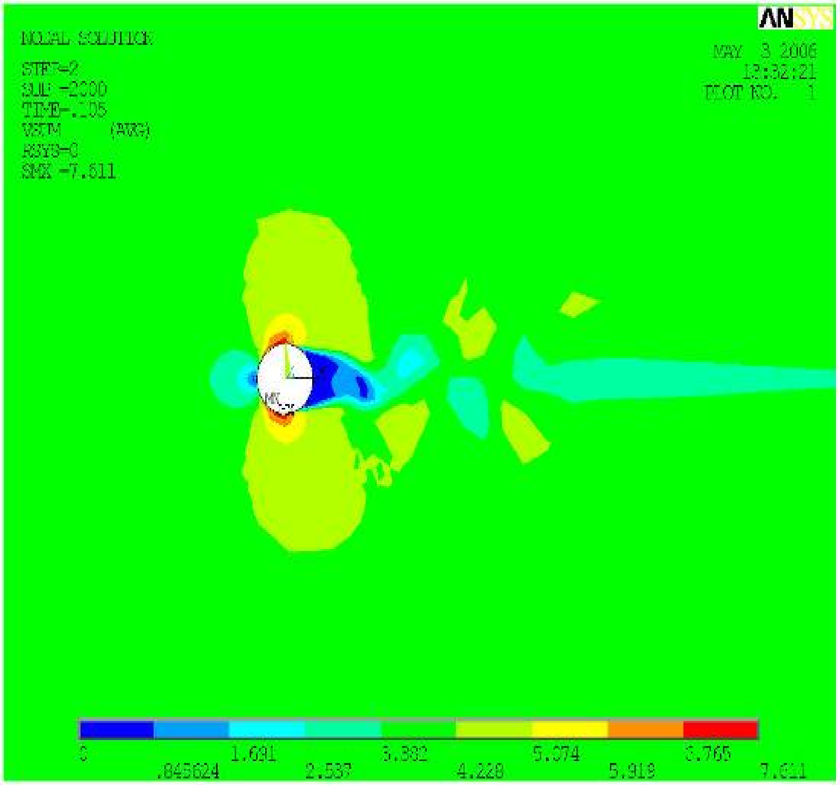
Dynamics

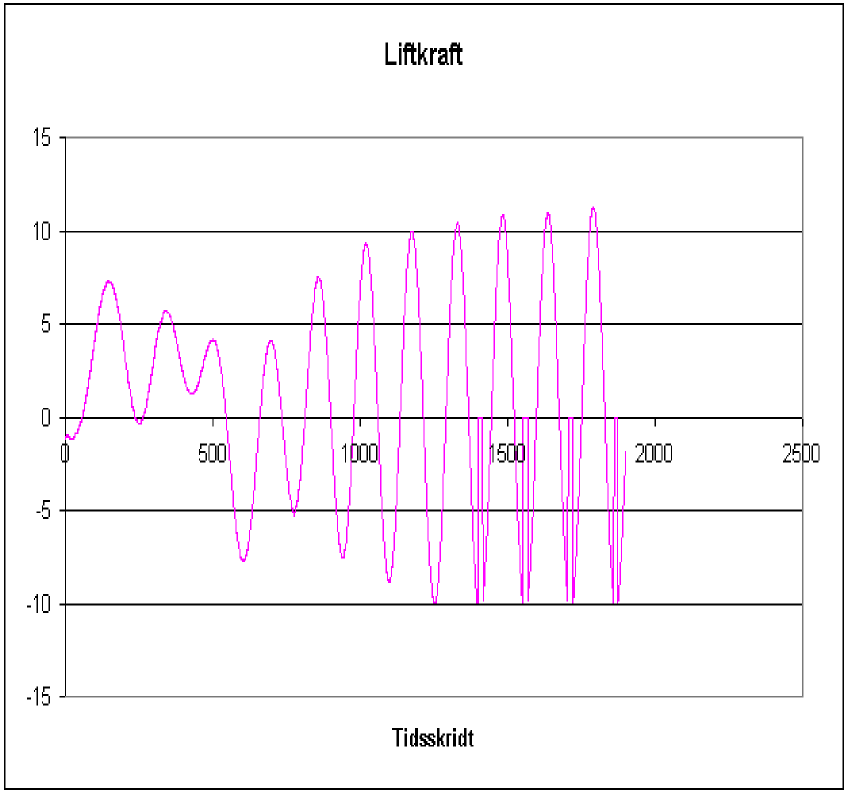
- The dynamic response need to be considered to accurately predict damage



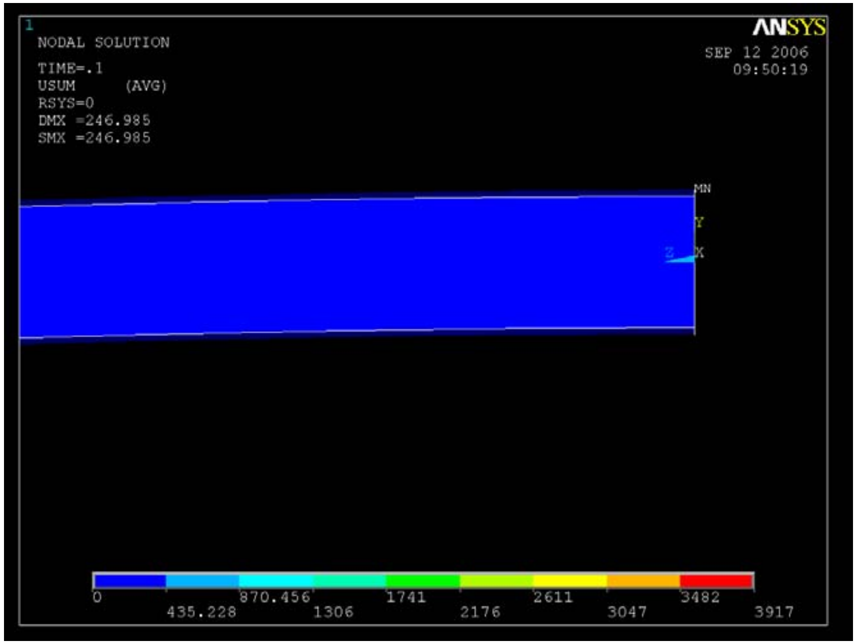


Dynamics/FSI





Composites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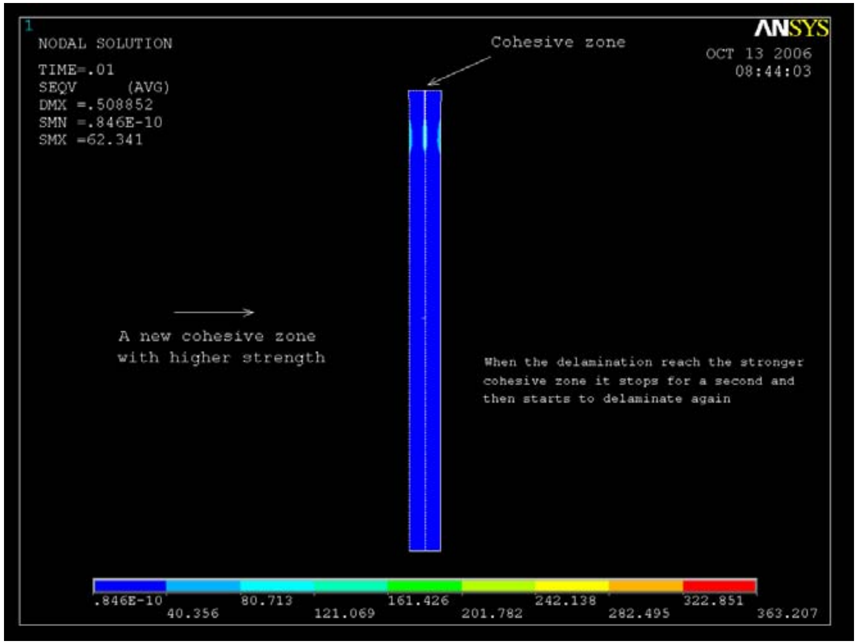
2611
3047
3482
3917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DMX =.508852
SMN =.846E-10
SMX =62.341
Cohesive zone
OCT 13 2006
08:44:03
A new cohesive zone
with higher strength
When the delamination reach the stronger
cohesive zone it stops for a second and
then starts to delaminate again
.846E-10
40.356
80.713
121.069
161.426
201.782
242.138
282.495
322.851
363.207

Definitions

- *Large strain*.
  - The strains themselves may be large, say over 5%. Examples: rubber structures (tires, membranes), metal forming. These are frequently associated with material nonlinearities.

- *Small strains but finite displacements and/or rotations*.
  - Slender structures undergoing finite displacements and rotations although the deformational strains may be treated as infinitesimal. Example: cables, springs, arches, bars, thin plates.

- *Linearized prebuckling*.
  - When both strains and displacements may be treated as infinitesimal before loss of stability by buckling. These may be viewed as initially stressed members. Example: many civil engineering structures such as buildings and stiff (non-suspended) bridges.

Definitions – large strain

- Large strain (or finite strain) - The shape change of the elements need to be taken into account (i.e., strains are finite).
- Rigid-body effects (e.g., large rotation) are also taken into account.
- An example is metal yielding. Note that ANSYS uses the term "large strain“.

- Finite strain implies that the strains are not infinitesimal, but a finite amount. (Sometimes users think that "large strain" is a lot of strain, but really, it's any case where there is a finite amount, not an excessively large amount, of strain.)

Definitions – large strains

- Small deflection and small strain analyses assume that displacements are small enough that the resulting stiffness changes are insignificant.
- In contrast, *large strain* analyses account for the stiffness changes that result from changes in an element's shape and orientation.

Definitions – large deflection

- Large deflection (or large rotation) - The strains are assumed to be small, but rigid-body effects (e.g., large rotation) are taken into account.
- An example is a long, slender fishing rod; when it bends due to the fish, each segment of the rod may not strain, but the total deformation may be large.
- Basically, this allows the actual strain to be "weeded out" from the displacements; it separates displacements due to rigid-body motion and those associated with the small strains.

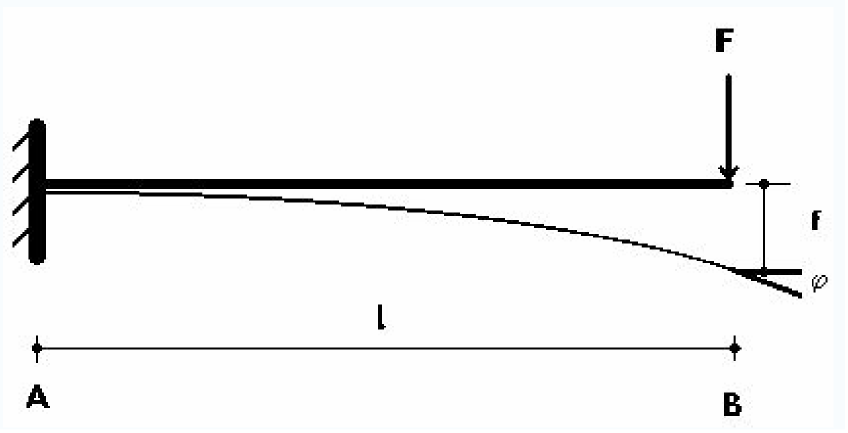
Definitions - deformation

- In engineering mechanics, **deformation** is a change in shape due to an applied force. This can be a result of tensile (pulling) forces, compressive (pushing) forces, shear, bending or torsion (twisting).
- In the figure it can be seen that the compressive loading (indicated by the arrow) has caused deformation in the cylinder so that the original shape (dashed lines) has changed (deformed) into one with bulging sides. The sides bulge because the material, although strong enough to not crack or otherwise fail, is not strong enough to support the load without change, thus the material is forced out laterally.

Definitions - deflection

Deflection (f) in engineering

- In engineering mechanics, **deflection** is a term to describe the degree to which a construction or structural element bends under a load. An example of the use of deflection in this context is in building construction. Architects and builders select materials for various applications. The beams used for frame work are selected on the basis of deflection, amongst other factors.



Definitions - deflection

- The elastic deflection f and angle of deflection φ (in radians) in the example image, a cantilever beam, can be calculated using:
  - fB = F·L3 / (3·E·I)
  - φB = F·L2 / (2·E·I)
- where
  - F = force acting on the tip of the beam
  - L = length of the beam (span)
  - E = modulus of elasticity
  - I = area moment of inertia
- From this formula it follows that the span L is the most determinating factor; if the span doubles, the deflection increases 23 = 8 fold.

- Building codes determine the maximum deflection, usually as a fraction of the span e.g. 1/400 or 1/600. Either (tensile) strength or deflection can determine the minimum dimension of the beam.

Definitions - Stress Stiffening

- The out-of-plane stiffness of a structure can be significantly affected by the state of in-plane stress in that structure. This coupling between in-plane stress and transverse stiffness, known as *stress stiffening*, is most pronounced in thin, highly stressed structures, such as cables or membranes.

- A drumhead, which gains lateral stiffness as it is tightened, would be a common example of a stress-stiffened structure.

Definitions - Stress Stiffening

- Even though stress stiffening theory assumes that an element's rotations and strains are small, in some structural systems (such as in Figure (a)), the stiffening stress is only obtainable by performing a large deflection analysis.

- In other systems (such as in Figure (b)), the stiffening stress is obtainable using small deflection, or linear, theory.