Lecture in Nonlinear FEM

on

the Building- and Civil Engineering sectors 8.th. semester

for

the Building- and Civil Engineering, B8k, and Mechanical Engineering, B8m

AALBORG UNIVERSITY ESBJERG, DENMARK

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Theme:
Design of marine constructions.
Outline: Updated: 15. februar 2005

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3. Geometrical nonlinearity - appl. in buckling analysis Cook 17.10
4. Stress stiffness Cook 18.1-18.4
5. Buckling Cook 18.5-18.6
6. Material nonlinearity - introduction Cook 17.3-17.4
7. Material nonlinearity - solution methods Cook 17.6, 17.2
8. Contact nonlinearity Cook 17.8
10. Nonlinear dynamic problems Cook 11.11-11.18

Literature:

Noter → A. Kristensen: http://www.aau.e.dk/bm/dk/notes.html

1. Introduction

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Nonlinear problems

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Assignments
Linear FEA revisited

Linear FEA is based on

- linearized geometrical equations (strain-displacement relations):
  \[ \{ \varepsilon \} = [B]\{d\} \]

- linearized constitutive equations (stress-strain relations):
  \[ \{ \sigma \} = [E]\{ \varepsilon \} = [E][B]\{d\} \]

- equations of equilibrium: \( \{R^i\} = \{R^e\} \), linear so that:
  \[ [K]\{D\} = \{R^e\} \]

and suitable boundary conditions, i.e. the assumptions made are often crude.
Motivation

Why perform nonlinear FEA? *Due to its ability to isolate primary stresses, in-elastic analysis uses the available load carrying capacity of a structure in the best way possible without compromising the overall safety of the structure against failure by excessive deformation. For many components, the use of in-elastic analysis results in a much more efficient design than what can be achieved with elastic methods or "design by rule".*

Other reasons for performing nonlinear FEA:

- In order to illustrate limitations to the linear analysis.
- In order to analyze time effects, e.g. crack growth, material properties, . . . ).
- In order to analyze causes of structural failure.
- In order to perform simulations, e.g. push-over analysis, crash tests. . . .
- In research
## Motivation

Linear versus nonlinear analysis [NAFEMS]:

<table>
<thead>
<tr>
<th>Feature</th>
<th>Linear problems</th>
<th>Nonlinear problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load-displacement relationship</td>
<td>Displacements are linearly dependent on the applied loads.</td>
<td>The load-displacement relationships are usually nonlinear.</td>
</tr>
<tr>
<td>Stress-strain relationship</td>
<td>A linear relationship is assumed between stress and strain.</td>
<td>In problems involving material nonlinearity, the stress-strain relationship is often a nonlinear function of stress, strain and/or time.</td>
</tr>
<tr>
<td>Magnitude of displacement</td>
<td>Changes in geometry due to displacement are assumed to be small and hence ignored, and the original (undeformed) state is always used as the reference state.</td>
<td>Displacements may not be small, hence an updated reference state may be needed.</td>
</tr>
<tr>
<td>Material properties</td>
<td>Linear elastic material properties are usually easy to obtain.</td>
<td>Nonlinear material properties may be difficult to obtain and may require additional experimental testing.</td>
</tr>
</tbody>
</table>
Motivation

Linear versus nonlinear analysis [NAFEMS]:

<table>
<thead>
<tr>
<th>Reversibility</th>
<th>The behaviour of the structure is completely reversible upon removal of the external loads.</th>
<th>Upon removal of the external loads, the final state may be different from the initial state.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary conditions</td>
<td>Boundary conditions remain unchanged throughout the analysis.</td>
<td>Boundary conditions may change, e.g. a change in the contact area.</td>
</tr>
<tr>
<td>Loading sequence</td>
<td>Loading sequence is not important, and the final state is unaffected by the load history.</td>
<td>The behaviour of the structure may depend on the load history.</td>
</tr>
<tr>
<td>Iterations and increments</td>
<td>The load is applied in one load step with no iterations.</td>
<td>The load is often divided into small increments with iterations performed to ensure that equilibrium is satisfied at every load increment.</td>
</tr>
<tr>
<td>Computation time</td>
<td>Computation time is relatively small in comparison to nonlinear problems.</td>
<td>Due to the many solution steps required for load incrementation and iterations, computation time is high, particularly if a high degree of accuracy is sought.</td>
</tr>
</tbody>
</table>
## Motivation

### Linear versus nonlinear analysis [NAFEMS]:

<table>
<thead>
<tr>
<th></th>
<th>Linear (Robustness)</th>
<th>Nonlinear (Initial state)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robustness of solutions</td>
<td>A solution can easily be obtained with no interaction from the user.</td>
<td>In difficult nonlinear problems, the FE code may fail to converge without some interaction from the user.</td>
</tr>
<tr>
<td>Use of results</td>
<td>Superposition and scaling allow results to be factored and combined as required.</td>
<td>Factoring and combining of results is not possible.</td>
</tr>
<tr>
<td>Initial state of stress/strain</td>
<td>The initial state of stress and/or strain is unimportant.</td>
<td>The initial state of stress and/or strain is usually required for material nonlinearity problems.</td>
</tr>
</tbody>
</table>
Nonlinear problems

Types of structural nonlinearity classifications used in engineering problems:

- Geometric nonlinearity
- Material nonlinearity:
  - time-independent behaviour such as plasticity
  - time-dependent behaviour such as creep
  - viscoelastic/viscoplastic behaviour where both plasticity and creep effects occur simultaneously
- Contact or boundary nonlinearity
Nonlinear problems

The term geometric nonlinearities models a number of physical problems:

Large strain: The strains themselves may be large, say over 5%. Examples: rubber structures (tires, membranes), metal forming. These are frequently associated with material nonlinearities.

Small strains but finite displacements and/or rotations: Slender structures undergoing finite displacements and rotations although the deformational strains may be treated as infinitesimal. Example: cables, springs, arches, bars, thin plates.

Linearized prebuckling: When both strains and displacements may be treated as infinitesimal before loss of stability by buckling. These may be viewed as initially stressed members. Example: many civil engineering structures such as buildings and stiff (non-suspended) bridges.
Geometric nonlinearity

Physical source: Change in geometry as the structure deforms is taken into account in setting up the strain-displacement and equilibrium equations.

Applications: Slender structures in aerospace, civil and mechanical engineering applications. Tensile structures such as cables and inflatable membranes. Metal and plastic forming. Stability analysis of all types.

Mathematical source: Strain-displacement equations:

\[ \varepsilon_{ij} = (v_{i,j} + v_{j,i} + v_{k,i}v_{k,j})/2 \]

(Lagrangian) which is nonlinear. Internal equilibrium equations:

\[ \sigma_{ji,j} + p_i = 0 \]

In the classical linear theory of elasticity, \( \sigma_{ji} = \sigma_{ij} \) but that is not necessarily true if geometric nonlinearities are considered.
Material nonlinearity

Physical source: Material behavior depends on current deformation state and possibly past history of the deformation. Other constitutive variables (prestress, temperature, time, moisture, electromagnetic fields, etc.) may be involved.

Applications: Structures undergoing nonlinear elasticity, plasticity, viscoelasticity, creep, or inelastic rate effects.

Mathematical source: The constitutive equations that relate stresses and strains. For a linear elastic material $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ reduces to:

$$\sigma_{ij} = \frac{E}{1+\nu} \left[ \varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon_{kk} \right]$$

where the tensor $C_{ijkl}$ contains elastic moduli $E$. If the material does not fit the elastic model, generalizations of this equation are necessary, and a whole branch of continuum mechanics is devoted to the formulation, study and validation of constitutive equations.
Material nonlinearity

The engineering significance of material nonlinearities varies greatly across disciplines. They seem to occur most often in civil engineering, that deals with inherently nonlinear materials such as concrete, soils and low-strength steel. In mechanical engineering creep and plasticity are most important, frequently occurring in combination with strain-rate and thermal effects. In aerospace engineering material nonlinearities are less important and tend to be local in nature (for example, cracking and "localization"failures of composite materials).

Material nonlinearities may give rise to very complex phenomena such as path dependence, hysteresis, localization, shakedown, fatigue, progressive failure. The detailed numerical simulation of these phenomena in three dimensions is still beyond the capabilities of the most powerful computers.
Force BC nonlinearity

Physical source: Applied forces depend on deformation.

Applications: The most important engineering application concerns pressure loads of fluids. These include hydrostatic loads on submerged or container structures; aerodynamic and hydrodynamic loads caused by the motion of aeroform and hydroform fluids (wind loads, wave loads, drag forces). Of more mathematical interest are gyroscopic and non-conservative follower forces, but these are of interest only in a limited class of problems, particularly in aerospace engineering.

Mathematical source: The applied forces (prescribed surface tractions and/or body forces) depend on the displacements the former being more important in practice.
Displacement BC nonlinearity

Physical source: Displacement boundary conditions depend on the deformation of the structure.

Applications: The most important application is the contact problem, i.e. contact-impact in dynamics, in which no-inter-penetration conditions are enforced on flexible bodies while the extent of the contact area is unknown. Nonstructural applications of this problem pertain to the more general class of free boundary problems, for example: ice melting, phase changes, flow in porous media. The determination of the essential boundary conditions is a key part of the solution process.

Mathematical source: For the contact problem: prescribed displacements depend on internal displacements. More complicated dependencies can occur in the free-boundary problems.
Response diagrams

A general load-deflection response diagram - an example.
Response diagrams

Fundamental (primary) and secondary equilibrium paths. Identification of critical, turning, and failure points.
Response diagrams

The response diagram for a purely linear structural model. Equilibrium points that are not critical are called regular.
Response diagrams

The linear response behaviour implies:

- A linear structure can sustain any load whatsoever and undergo any displacement magnitude.
- There are no critical, turning or failure points.
- Response to different load systems can be obtained by superposition.
- Removing all loads returns the structure to the reference position.
Response diagrams

Thus following assumptions have been made:

- Elastic behaviour: Linear elastic homogeneous isotropic material
- Linearization: Small displacements ⇒ translations, rotations and deflections are small
- Normality: Plane sections remains plane (small rotations) ⇒ the strain in each fiber dependent linearly of the distance to the neutral axis. Hooke’s law applies to each fiber
- Each cross-section remains constant

This idealization allow the principle of superposition to be applied. Furthermore linear analysis allow all loads to be applied instantaneously and the loading history is irrelevant, i.e. displacements are linearly dependent on the loads and the solution can be scaled.
Response diagrams

A control-state response diagram.

Control parameter $\lambda$

Equilibrium path

State parameter $\mu$ or $u$
Response diagrams

Basic types of nonlinear response: (a) Linear until brittle failure, (b) Stiffening or hardening, (c) Softening.
Response diagrams

a) This response is characteristic for pure crystals, glassy, and certain high strength composite materials.

b) This response is typical for cables, pneumatic (inflatable) structures, which may be collectively called tensile structures. The stiffening effect comes from geometry "adaptation" to the applied loads. Some flat-plate assemblies also display this behavior initially.

c) This response is more common for structural materials than the previous two. A linear response is followed by a softening regime that may occur slowly or suddenly.

* Here B and T denote bifurcation and turning points, respectively.
Response diagrams

Examples on complex response patterns: (d) snap-through, (e) snap-back, (f) bifurcation, (g) bifurcation combined with limit points and snap-back.
Response diagrams

d) This snap-through response combines softening with hardening following the second limit point. The response branch between the two limit points has a negative stiffness and is therefore unstable. If the structure is subject to a prescribed constant load, the structure "takes off" dynamically when the first limit point is reached. A response of this type is typical for slightly curved structures such as shallow arches.

e) This snap-back response is an exaggerated snap-through, in which the response curve "turns back" in itself with the consequent appearance of turning points. The equilibrium between the two turning points may be stable and consequently physically realizable. This type of response is exhibited by trussed-dome, folded and thin-shell structures in which "moving arch" effects occur following the first limit point; for example cylindrical shells with free edges and supported by end diaphragms.
Response diagrams

In all previous diagrams the response was a unique curve. The presence of bifurcation/buckling points as in f) and g) introduces more features. At such points more than one response path is possible. The structure takes the path that is dynamically preferred, i.e. having a lower energy, over the others. Bifurcation points may occur in any sufficiently thin structure that experiences compressive stresses.

An example of such a complicated response is provided by thin cylindrical shells under axial compression.