Course in Nonlinear FEM

Material nonlinearity II
Outline

Lecture 1 – Introduction
Lecture 2 – Geometric nonlinearity
Lecture 3 – Material nonlinearity
Lecture 4 – Material nonlinearity continued
Lecture 5 – Geometric nonlinearity revisited
Lecture 6 – Issues in nonlinear FEA
Lecture 7 – Contact nonlinearity
Lecture 8 – Contact nonlinearity continued
Lecture 9 – Dynamics
Lecture 10 – Dynamics continued
Nonlinear FEM

Lecture 1 – Introduction, Cook [17.1]:
- Types of nonlinear problems
- Definitions

Lecture 2 – Geometric nonlinearity, Cook [17.10, 18.1-18.6]:
- Linear buckling or eigen buckling
- Prestress and stress stiffening
- Nonlinear buckling and imperfections
- Solution methods

Lecture 3 – Material nonlinearity, Cook [17.3, 17.4]:
- Plasticity systems
- Yield criteria

Lecture 4 – Material nonlinearity revisited, Cook [17.6, 17.2]:
- Flow rules
- Hardening rules
- Tangent stiffness
Nonlinear FEM

Lecture 5 – Geometric nonlinearity revisited, Cook [17.9, 17.3-17.4]:
- The incremental equation of equilibrium
- The nonlinear strain-displacement matrix
- The tangent-stiffness matrix
- Strain measures

Lecture 6 – Issues in nonlinear FEA, Cook [17.2, 17.9-17.10]:
- Solution methods and strategies
- Convergence and stop criteria
- Postprocessing/Results
- Troubleshooting
Nonlinear FEM

Lecture 7 – Contact nonlinearity, Cook [17.8]:
– Contact applications
– Contact kinematics
– Contact algorithms

Lecture 8 – Contact nonlinearity continued, Cook [17.8]:
– Issues in FE contact analysis/troubleshooting

Lecture 9 – Dynamics, Cook [11.1-11.5]:
– Solution methods
– Implicit methods
– Explicit methods

Lecture 10 – Dynamics continued, Cook [11.11-11.18]:
– Dynamic problems and models
– Damping
– Issues in FE dynamic analysis/troubleshooting
References

• **[ANSYS]** ANSYS 10.0 Documentation (installed with ANSYS):
  – Basic Analysis Procedures
  – Advanced Analysis Techniques
  – Modeling and Meshing Guide
  – Structural Analysis Guide
  – Thermal Analysis Guide
  – APDL Programmer’s Guide
  – ANSYS Tutorials

• **[Cook]** Cook, R. D.; Concepts and applications of finite element analysis, John Wiley & Sons
Plasticity Theory

1. Yield Criterion
2. Flow Rule
3. Hardening Rule
Flow Rule

Define a plastic potential $Q$, which is a function of stresses $\{\sigma\}$ and parameters $\{\alpha\}$ and $W_p$ associated with the hardening rule. Also define a scalar $d\lambda$ that may be called a “plastic multiplier.” Plastic strain increments are given by:

$$Q = Q(\{\sigma\}, \{\alpha\}, W_p)$$

$$\{d\varepsilon^p\} = \left\{ \frac{\partial Q}{\partial \{\sigma\}} \right\} d\lambda$$
Flow Rule

\[ d\varepsilon^p_x = \frac{\partial Q}{\partial \sigma_x} \, d\lambda \]

\[ d\varepsilon^p_y = \frac{\partial Q}{\partial \sigma_y} \, d\lambda \]

\[ \vdots \]

\[ d\gamma^p_{xz} = \frac{\partial Q}{\partial \sigma_{xz}} \, d\lambda \]
Flow Rule

\[ Q = F \quad \text{associated flow rule.} \]
\[ Q \neq F \quad \text{nonassociated flow rule.} \]

*associated flow rule - ductile materials*

*nonassociated flow rule - granular materials (soils)*
Flow Rule

1. Relates stress increment \( \{ d\sigma \} \) to strain increment \( \{ d\varepsilon \} \) after yielding.
2. Uniaxial case: \( d\sigma = E_t \ d\varepsilon \)
3. Prandtl-Reuss often used.
5. Non-associated - soil or granular materials.
Hardening Rule

- If an unloading is followed by a reversed loading, e.g. tension is followed by compression, metals exhibit yielding at lower load than was the original yield limit. This is termed the Bauschinger effect.

- In a general multi-axial stress state, the hardening phenomena correspond to change in the size/shape and/or translation of the original elastic domain. This phenomenon is often simplified by assuming that the elastic domain does not change in shape, but only uniformly expands (isotropic hardening) or translates (kinematic hardening) or expands and translates (mixed hardening) in the stress space.
Hardening Rule

Basically two hardening rules exist:

- isotropic hardening $F = |\sigma| - \sigma_0$ - Bauschinger effect is ignored but the elastic range expands
- kinematic hardening $F = |\sigma - \alpha| - \sigma_Y$ - Bauschinger effect is included but elastic range remains constant
Hardening Rule

The parameter $\{\alpha\}$ locates the center of the yield surface in stress space. Before any yielding occurs $\{\alpha\} = 0$. In kinematic hardening the yield surface moves in the direction of plastic straining, so $\{\alpha\} \neq 0$. The parameter $W_p$ describes how the yield surface grows. For isotropic hardening, $\{\alpha\} = 0$ throughout the analysis and $W_p \neq 0$. 
Hardening Rule

The parameters \( \{\alpha\} \) and \( W_p \) are given by:

\[
\{\alpha\} = \int C \{d\varepsilon^p\}
\]

\[
W_p = \int \{\sigma\}^T \{d\varepsilon^p\}
\]

Where C can be assumed to be a material constant.
Hardening Rules

1. **Kinematic**
   1. Yield surface retains size and shape and translates in stress space.

2. **Isotropic**
   1. Yield surface retains shape but increases in size.
A diagram illustrating the material nonlinearity of a structure. The graph shows the relationship between stress (\(\sigma\)) and strain (\(\varepsilon\)). The diagram includes labels for different stress levels, such as \(\sigma_B\), \(\sigma_Y\), and the strain at fracture. The terms 'Kinematic' and 'Isotropic' are used to describe the behavior of the material under different conditions.
Yield Surface

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \sigma_3 \]
Isotropic Hardening

Yield Surface

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \sigma_3 \]
Kinematic Hardening

Yield Surface

σ₁

σ₂

σ₃
Material Properties

Nonlinear Material Properties

- Nonlinear material properties are usually tabular data, such as plasticity data (stress-strain curves for different hardening laws), magnetic field data (B-H curves), creep data, swelling data, hyperelastic material data, etc.

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Material Properties

• The automatic time stepping feature [AUTOTS] will respond to plasticity after the fact, by reducing the load step size after a load step in which a large number of equilibrium iterations was performed or in which a plastic strain increment greater than 15% was encountered. If too large a step was taken, the program will bisect and resolve using a smaller step size.
Material Properties

Nonlinear Material Properties

- Other kinds of nonlinear behavior might also occur along with plasticity. In particular, large deflection and large strain geometric nonlinearities will often be associated with plastic material response. If you expect large deformations in your structure, you must activate these effects in your analysis with the `NLGEOM` command.

- For large strain analyses, material stress-strain properties must be input in terms of *true* stress and *logarithmic* strain.
Material Properties

Nonlinear Material Properties

- Plastic Material Options

  - **Bilinear Kinematic Hardening (BKITN)**
    - option assumes the total stress range is equal to twice the yield stress, so that the Bauschinger effect is included

  - **Multilinear Kinematic Hardening (KINH and MKIN)**
    - use the Besseling model, also called the sublayer or overlay model, so that the Bauschinger effect is included
    - KINH uses Rice's model where the total plastic strains remain constant by scaling the sublayers

- **Nonlinear Kinematic Hardening (CHABOCHE)**
  - a multi-component nonlinear kinematic hardening model that allows you to superpose several kinematic models
Material Properties

Nonlinear Material Properties

- Plastic Material Options
  - **Bilinear Isotropic Hardening** (BISO)
    - uses the von Mises yield criteria coupled with an isotropic work hardening assumption
    - for large strain analyses
  - **Multilinear Isotropic Hardening** (MISO)
    - for large strain analyses
  - **Nonlinear Isotropic Hardening** (NLISO)
    - based on the Voce hardening law
  - **Anisotropic** (ANISO)
    - allows for different bilinear stress-strain behavior in the material x, y, and z directions as well as different behavior in tension, compression, and shear. This option is applicable to metals that have undergone some previous deformation (such as rolling)
  - **Hill Anisotropy** (HILL)
    - combined with other material options simulates plasticity, viscoplasticity, and creep - all using the Hill potential
  - **Drucker-Prager** (DP)
    - applicable to granular (frictional) material such as soils, rock, and concrete, and uses the outer cone approximation to the Mohr-Coulomb law
1-D Elastic-Plastic Analysis

\[ \sigma = E \varepsilon \]

\[ \varepsilon^e = \frac{\sigma_B}{E} \]
Elastic-Plastic Action in Uniaxial Tension

1. Stress reaches yield value (onset of yielding).
2. Subsequent plastic deformation may alter the stress needed to produce renewed or continued yielding.
3. If $E_t > 0$ this stress will exceed $\sigma_y$.
4. Flow Rule: $d\sigma = E_t \, d\varepsilon$.
5. Prior to onset of yield or during unloading: $d\sigma = E \, d\varepsilon$.
6. Complete unloading from B to C results in permanent strain $\varepsilon^p$.
7. Behavior does not have to be bilinear. $E_t$ need not be constant.
Plastic Flow

1. Yielding has occurred.
2. Strain increment $d\varepsilon$ takes place.
3. $d\varepsilon = d\varepsilon_e + d\varepsilon_p$
4. Write stress increment in various ways:

$$d\sigma = E(d\varepsilon - d\varepsilon_p)$$
$$d\sigma = E_t d\varepsilon$$
$$d\sigma = H d\varepsilon_p$$
Material nonlinearity II

\[
d\sigma = E(d\varepsilon - d\varepsilon^p)
\]
\[
d\sigma = E_t d\varepsilon
\]
\[
d\sigma = H d\varepsilon^p
\]
\[
H = \left( \frac{E_t}{1 - (E_t/E)} \right)
\]
\[
E_t = E \left( 1 - \frac{E}{E + H} \right)
\]
\[ E = \frac{d\sigma}{d\varepsilon} \quad |\sigma| < \sigma_Y \]

\[ E_t = \frac{d\sigma}{d\varepsilon} \quad \text{after yielding} \]
\[ [k_t] = \frac{AE_{ep}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ E_{ep} = E \quad \text{before yielding} \]

\[ \text{or during unloading} \]

\[ E_{ep} = E_t \quad \text{after yielding} \]
Rounding the corner - going from $\varepsilon_D$ to $\varepsilon_A$:

$$E_{ep} = mE + (1 - m)E_t$$

$$m = \frac{\varepsilon_Y - \varepsilon_D}{\varepsilon_A - \varepsilon_D}$$

or

$$\sigma^* = E \varepsilon_A$$

$$m = \frac{\sigma_Y - \sigma_D}{\sigma^* - \sigma_D}$$
Tangent Stiffness Method

1. For the first computational cycle \( (i = 1) \), assume \( E_{ep} = E \) for all elements. Apply the first load increment \( \{\Delta R\}_1 \)

2. Using the current strains, determine the current \( E_{ep} \) in each element. Obtain the \( [k_t]_n \) for each element \( n \). Obtain the current structure tangent stiffness \( [K_t]_{i-1} \). Solve \( [K_t]_{i-1} \{\Delta D\}_i = \{\Delta R\}_i \). From \( \{\Delta D\}_i \) obtain the current strain increment \( \Delta \varepsilon_i \) for each element
Tangent Stiffness Method

3. If any element makes the elastic-to-plastic transition revise $E_{ep}$ Return to previous step 2 and repeat 2 & 3 until convergence. Define convergence as $\Delta \varepsilon < (\_\%) \varepsilon$.

4. Update $\{D\}_i = \{D\}_{i-1} + \{\Delta D\}_i$, $\varepsilon_i = \varepsilon_{i-1} + \Delta \varepsilon_i \sigma_i = \sigma_{i-1} + \Delta \sigma_i$.

5. Apply next load increment and return to step 2.

6. Stop when sum of incremental loads equals the total load.
$P_1 = \Delta P_1$

$D_1 = \Delta D_1$

$\Delta P_2$

$\Delta P_3$

$\Delta D_2$

$\Delta D_3$

EXACT
Improve Results

1. Smaller load increments
2. Exercising Step 3
3. Use Corrective Loads
4. Can attempt to choose load increments so only one (or a few) elements yield in each load step.
Initial Stiffness Method

1. Avoids having to formulate tangent stiffness matrix for each load increment.
2. Can converge slowly if plastic strains are large or widespread.
We seek the strain $\varepsilon_B$ associated with $\sigma_B$

$$\varepsilon_C = \frac{\sigma_C}{E}$$

$$\sigma_C = \sigma_B + E\Delta\varepsilon^p$$

$$\Delta\varepsilon^p = \frac{1}{H} \Delta\sigma = \frac{1}{H} E_t \Delta\varepsilon$$

$$\Delta\varepsilon^p = \frac{E}{E + H} \Delta\varepsilon = \left(1 - \frac{E_t}{E}\right) \Delta\varepsilon$$
One Dimensional Elastic-Plastic Analysis

\[ \Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p \]
Tangent Stiffness Method

1. Compute the elastic stiffness matrix \([K]\). Solve \([K]\{D\} = \{R\}\) where \(\{R\}\) is proportional to the actual loads but of arbitrary value. Scale \(\{R\}\) so that it becomes \(\{R_Y\}\) which causes yielding to impend. Scale \(\{D\}\) and call it \(\{D_{old}\}\). Choose subsequent load increments to be greater of 0.5\(\{R_Y\}\) or \((E_t/E)\ \{R_Y\}\). Initialize supplementary loads to \(\{\Delta R_s\}\) =0.

2. Solve \([K]\{\Delta D\} = \{\Delta R\} + \{\Delta R_s\}\) for \(\{\Delta D\}\)
Tangent Stiffness Method

3. Update displacements: \( \{D\}_{\text{new}} = \{D\}_{\text{old}} + \{\Delta D\} \)

4. In each element, calculate the strain increment \( \Delta \varepsilon \) associated with \( \{\Delta D\} \). Update element stresses by adding \( \Delta \sigma \) to existing stress \( \sigma \), using \( \Delta \sigma = E \Delta \varepsilon \) if \( \sigma < \sigma_Y \) and \( \Delta \sigma = E_t \Delta \varepsilon \) if \( \sigma > \sigma_Y \). For elements that make the elastic-to-plastic transition by the addition of \( \Delta \sigma \), evaluate \( m \) and recompute \( \Delta \sigma \) as \( \Delta \sigma = E m \Delta \varepsilon \).
Tangent Stiffness Method

5. For all elements that display plastic strains ($|\sigma| > \sigma_Y$), calculate the plastic strain increments by $\Delta \varepsilon^p = (1-E_t/E) \Delta \varepsilon$ or $\Delta \varepsilon^p = (1-E_t/E) (1-m) \Delta \varepsilon$ for elements making the elastic-to-plastic transition. Generate the supplementary loads by summing element contributions:

$$\{\Delta R_s \} = \sum \{\Delta r_s \}$$

$$\{\Delta r_s \} = \int_0^L [B]^T E \Delta \varepsilon^p A \, dx$$
Tangent Stiffness Method

6. (continued) Solve \([K]\{\Delta D\} = \{\Delta R_s\}\) and return to step 3.

7. Repeat steps 3 to 6 until convergence. Then apply another load increment \({\Delta R}\) and return to step 2

8. Stop when \({R_Y}\) + \(\sum{\Delta R}\) reaches the total load.
Incremental Stress-Strain Relation

\[
dF = 0 = \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T \{d\sigma\} + \left\{ \frac{\partial F}{\partial \{\alpha\}} \right\}^T \{d\alpha\} + \frac{\partial F}{\partial W_p} dW_p
\]

\[
\{d\alpha\} = C \{d\varepsilon^p\}
\]

\[
dW_p = \{\sigma\}^T \{d\varepsilon_p\}
\]

\[
\{d\sigma\} = [E] \{d\varepsilon^e\} = [E] (\{d\varepsilon\} - \{d\varepsilon^p\})
\]
Incremental Stress-Strain Relation

\[ d\lambda = \{C_\lambda\}^T \{d\varepsilon\} \]

\[
\{C_\lambda\}^T = \frac{\begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \\ \frac{\partial F}{\partial \{\alpha\}} \end{bmatrix}^T [E] \begin{bmatrix} \frac{\partial Q}{\partial \{\sigma\}} \\ \frac{\partial Q}{\partial \{\alpha\}} \end{bmatrix} - C \begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \\ \frac{\partial F}{\partial \{\alpha\}} \end{bmatrix}^T \begin{bmatrix} \frac{\partial Q}{\partial \{\sigma\}} \\ \frac{\partial Q}{\partial \{\alpha\}} \end{bmatrix} - \frac{\partial F}{\partial W_p} \begin{bmatrix} \frac{\partial Q}{\partial \{\sigma\}} \end{bmatrix}^T \begin{bmatrix} \frac{\partial Q}{\partial \{\sigma\}} \end{bmatrix}} {\begin{bmatrix} \frac{\partial F}{\partial \{\sigma\}} \\ \frac{\partial F}{\partial \{\alpha\}} \end{bmatrix}^T [E] \begin{bmatrix} \frac{\partial Q}{\partial \{\sigma\}} \\ \frac{\partial Q}{\partial \{\alpha\}} \end{bmatrix}}
\]
Incremental Stress-Strain Relation

\[
\{d\sigma\} = [E]\left(\{d\varepsilon\} - \left\{ \frac{\partial Q}{\partial \{\sigma\}} \right\}d\lambda \right)
\]

\[
\{d\sigma\} = [E_{ep}]{d\varepsilon}
\]

\[
[E_{ep}] = [E] - \left\{ \frac{\partial Q}{\partial \{\sigma\}} \right\}\{C_\lambda\}^T
\]

\[
[k_t] = \int_V [B]^T [E_{ep}][B]dV
\]
\[ F = 0 \quad \text{and} \quad dF = 0 \]

\[
\begin{bmatrix}
E_{ep} \\
\end{bmatrix} = 
\begin{bmatrix}
E \\
\end{bmatrix} - 
\left\{ \frac{\partial Q}{\partial \{\sigma\}} \right\} \{C_\lambda \}^T
\]

\[ F < 0 \quad \text{and} \quad dF < 0 \]

\[
\begin{bmatrix}
E_{ep} \\
\end{bmatrix} = 
\begin{bmatrix}
E \\
\end{bmatrix}
\]