## Kapitel 2

## Orthotropic Materials

### 2.1 Elastic Strain matrix

Elastic strains are related to stresses by Hooke's law, as stated below. The stressstrain relationship is in each material point formulated in the local cartesian coordinate system.

$$
\begin{equation*}
\varepsilon_{e}=\mathbf{C} \boldsymbol{\sigma} \tag{1}
\end{equation*}
$$

$\boldsymbol{\sigma}$ is the stress vector, which uses the same convention as the strain vector. The stress vector is given below.

$$
\begin{equation*}
\boldsymbol{\sigma}=\left[\sigma_{l} \sigma_{r} \sigma_{t} \tau_{l r} \tau_{l t} \tau_{r t}\right]^{\mathrm{T}} \tag{2}
\end{equation*}
$$

$\mathbf{C}$ is the material compliance matrix. The compliance matrix is given as

$$
\mathbf{C}=\left[\begin{array}{cccccc}
\frac{1}{E_{l}} & -\frac{\nu_{r l}}{E_{r}} & -\frac{\nu_{t l}}{E_{t}} & 0 & 0 & 0  \tag{3}\\
-\frac{\nu_{l n}}{E_{l}} & \frac{1}{E_{r}} & -\frac{\nu_{t r}}{E_{t}} & 0 & 0 & 0 \\
-\frac{\nu}{l_{l t}} & -\frac{\nu_{\nu_{r t}}}{E_{r}} & \frac{1}{E_{t}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{l r}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{l t}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{r t}}
\end{array}\right]
$$

E's are moduli of elasticity, $G$ 's are shear moduli and $\nu$ 's are Poissons ratios. The compliance matrix should be symmetric, which gives rise to the following restric-
tions:

$$
\begin{equation*}
\nu_{r l}=\nu_{l r} \frac{E_{r}}{E_{l}} \quad ; \quad \nu_{t l}=\nu_{l t} \frac{E_{t}}{E_{l}} \quad ; \quad \nu_{t r}=\nu_{r t} \frac{E_{t}}{E_{r}} \tag{4}
\end{equation*}
$$

The inverse of the compliance matrix is the material stiffness matrix, $\mathbf{D}$, which give the stresses produced by an elastic strain state.
$\mathbf{D}=\left[\begin{array}{cccccc}E_{l}\left(1-\nu_{r t} \nu_{t r}\right) / k & E_{l}\left(\nu_{r t} \nu_{t l}+\nu_{r l}\right) / k & E_{l}\left(\nu_{r l} \nu_{t r}+\nu_{t l}\right) / k & 0 & 0 & 0 \\ E_{r}\left(\nu_{l t} \nu_{t r}+\nu_{l r}\right) / k & E_{r}\left(1-\nu_{l t} \nu_{t l}\right) / k & E_{r}\left(\nu_{l r} \nu_{t l}+\nu_{t r}\right) / k & 0 & 0 & 0 \\ E_{t}\left(\nu_{l r} \nu_{r t}+\nu_{l t}\right) / k & E_{t}\left(\nu_{l t} \nu_{r l}+\nu_{r t}\right) / k & E_{t}\left(1-\nu_{l r} \nu_{r l}\right) / k & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{l r} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{l t} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{r t}\end{array}\right]$
where $k=1-\nu_{r t} \nu_{t r}-\nu_{l r} \nu_{r l}-\nu_{l t} \nu_{t l}-\nu_{l r} \nu_{r t} \nu_{t l}-\nu_{r l} \nu_{t r} \nu_{l t}$. The stiffness matrix is also symmetric, which follows from the restrictions given in (4). Furthermore $\mathbf{D}$ should be positive definit i.e. $\boldsymbol{\varepsilon}^{T} \mathbf{D} \boldsymbol{\varepsilon}>0$ (The material resists deformations), which leads to the following restrictions:

$$
\begin{align*}
& 1-\nu_{r t} \nu_{t r}-\nu_{l r} \nu_{r l}-\nu_{l t} \nu_{t l}-\nu_{l r} \nu_{r t} \nu_{t l}-\nu_{r l} \nu_{t r} \nu_{l t}>0 \\
& 1-\nu_{r t} \nu_{t r}>0 \\
& 1-\nu_{l t} \nu_{t l}>0  \tag{6}\\
& 1-\nu_{l r} \nu_{r l}>0
\end{align*}
$$

The material stiffness matrix is used when the equilibrium equations are solved, using the Finite element method.

### 2.2 Coordinate systems

Three different cartesian coordinate systems are used in the spatial discretization, one global and two different local associated with each element. The local element coordinate systems are called element coordinate system and material coordinate system, respectively. The three different coordinate systems are shown in figure 1

### 2.2.1 Global coordinate system

The global coordinates are termed $(X, Y, Z)$. The origin is located in global node number 1. The Global coordinate system is used to define the geometry of the


Figur 1: Local and global coordinate systems
timber beam, and the boundary conditions.

### 2.2.2 Element coordinate system

The first of the local coordinate systems is called the element coordinate system. The coordinate axes are termed $(x, y, z)$. The origin is located in the center of the element. The positive direction for the $x$-axis is from node 2 towards node 1 . The positive direction for the $y$-axis is from node 4 towards node 1 and finally the positive direction for the $z$-axis is from node 3 towards node 1 . The element coordinate system is used for the interpolation between the element nodes and the element integration.

### 2.2.3 Material coordinate system

The second coordinate system is the element material coordinate system, with axes $(l, r, t)$. The origin of the material coordinate system is also located in the center of the element, and the material directions are assumed to be constant in the element, which make it necessary with a large number of elements in plane perpendicular to the grain direction.

### 2.2.4 Transformation between coordinate systems

The transformation of properties described in the material coordinate system to the element coordinate system is described in this section. Properties can be geometric points, material properties eg. material stiffness or physical quantities eg. stress or heat flux. The transformation of geometrical points between the two coordinate systems are:

$$
\left[\begin{array}{l}
x  \tag{7}\\
y \\
z
\end{array}\right]=\mathbf{A}^{\mathrm{T}}\left[\begin{array}{l}
l \\
r \\
t
\end{array}\right]
$$

Where $\mathbf{A}$ is given by (8)

$$
\mathbf{A}=\left[\begin{array}{lll}
l_{x} & l_{y} & l_{z}  \tag{8}\\
r_{x} & r_{y} & r_{z} \\
t_{x} & t_{y} & t_{z}
\end{array}\right]
$$

The first row in $\mathbf{A}$ is a unit vector in the $l$-direction, described in the $x y z$-coordinate system, the second is a unit vector in the $r$-direction and the last row is a unit vector in the $t$-direction. The elements in $\mathbf{A}$ are denoted direction cosines. The determination of direction cosines is given i section 2.2.5. A is often called a rotation matrix, and is an orthogonal matrix; i.e. $\mathbf{A}^{-1}=\mathbf{A}^{\mathrm{T}}$. Hence the inverse of the transformation in (7) is given below

$$
\left[\begin{array}{l}
l  \tag{9}\\
r \\
t
\end{array}\right]=\mathbf{A}\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]
$$

The transformation of stresses and strains, from one coordinate system to another is as given in (10)

$$
\begin{align*}
\boldsymbol{\varepsilon}_{l r t} & =\mathbf{G} \boldsymbol{\varepsilon}_{x y z} \\
\boldsymbol{\sigma}_{x y z} & =\mathbf{G}^{\mathrm{T}} \boldsymbol{\sigma}_{l r t}  \tag{10}\\
\boldsymbol{\sigma}_{l r t} & =\left(\mathbf{G}^{\mathrm{T}}\right)^{-1} \boldsymbol{\sigma}_{x y z}
\end{align*}
$$

where $\boldsymbol{\sigma}_{l r t}=\left[\sigma_{l} \sigma_{r} \sigma_{t} \tau_{l r} \tau_{l t} \tau_{r t}\right]^{\mathrm{T}}$ is the stress vector, described in the material coordinate system and $\boldsymbol{\sigma}_{x y z}$ is the stress vector described in the element coordinate
system. The transformation matrix $\mathbf{G}$ can be deduced from the elements in (8), and is as given here:

$$
\mathbf{G}=\left[\begin{array}{cccccc}
l_{x}^{2} & l_{y}^{2} & l_{z}^{2} & l_{x} l_{y} & l_{x} l_{z} & l_{y} l_{z}  \tag{11}\\
r_{x}^{2} & r_{y}^{2} & r_{z}^{2} & r_{x} r_{y} & r_{x} r_{z} & r_{y} r_{z} \\
t_{x}^{2} & t_{y}^{2} & t_{z}^{2} & t_{x} t_{y} & t_{x} t_{z} & t_{y} t_{z} \\
2 l_{x} r_{x} & 2 l_{y} r_{y} & 2 l_{z} r_{z} & l_{x} r_{y}+l_{y} r_{x} & l_{z} r_{x}+l_{x} r_{z} & l_{y} r_{z}+l_{z} r_{y} \\
2 l_{x} t_{x} & 2 l_{y} t_{y} & 2 l_{z} t_{z} & t_{x} l_{y}+t_{y} l_{x} & t_{z} l_{x}+t_{x} l_{z} & t_{y} l_{z}+t_{z} l_{y} \\
2 r_{x} t_{x} & 2 r_{y} t_{y} & 2 r_{z} t_{z} & r_{x} t_{y}+r_{y} t_{x} & r_{z} t_{x}+r_{x} t_{z} & r_{y} t_{z}+r_{z} t_{y}
\end{array}\right]
$$

The $\left(\mathbf{G}^{T}\right)^{-1}$ matrix is given in (12), when the $\mathbf{G}$ matrix is subdivided into four $3 \times 3$ matrices, termed $\mathbf{G}_{11}, \mathbf{G}_{12}, \mathbf{G}_{21}$ and $\mathbf{G}_{22}$.

$$
\mathbf{G}=\left[\begin{array}{ll}
\mathbf{G}_{11} & \mathbf{G}_{12}  \tag{12}\\
\mathbf{G}_{21} & \mathbf{G}_{22}
\end{array}\right] \quad, \quad \mathbf{G}^{-1}=\left[\begin{array}{cc}
\mathbf{G}_{11} & 2 \mathbf{G}_{12} \\
\frac{1}{2} \mathbf{G}_{21} & \mathbf{G}_{22}
\end{array}\right]^{T}
$$

The transformation of stiffness or flexibility properties, from the material to the element coordinate system, is performed by a tensor-like transformation, as stated below.

$$
\begin{equation*}
\mathbf{D}_{x y z}=\mathbf{G}^{\mathrm{T}} \mathbf{D}_{l r t} \mathbf{G} \tag{13}
\end{equation*}
$$

where $\mathbf{D}_{l r t}$ is the material stiffness matrix, formulated in the material coordinate system, given by (5) and $\mathbf{D}_{x y z}$ is the material stiffness matrix formulated in the element coordinate system. The $\mathbf{G}$ matrix is the transformation matrix given in (11) The transformation of thermal conductivities and moisture transport properties is done in a similar tensor-like way, as stated below

$$
\begin{equation*}
\boldsymbol{\lambda}_{x y z}=\mathbf{A}^{\mathrm{T}} \boldsymbol{\lambda}_{l r t} \mathbf{A} \tag{14}
\end{equation*}
$$

where $\boldsymbol{\lambda}_{l r t}$ is the material conductivity matrix formulated in the material coordinate system, which is a 3 by 3 diagonal matrix containing the thermal conductivities in the principal directions. $\boldsymbol{\lambda}_{x y z}$ is the material conductivity matrix formulated in the element coordinate system. $\mathbf{A}$ is the transformation matrix given in (8).

### 2.2.5 Direction Cosines

The direction cosines are determined from the different elements location relative to the pith, see figure 1, which are termed $\mathbf{A}_{0} . \mathbf{A}_{0}$ has the same structure as the matrix given in (8). Further the directions cosines are depended on two growth phenomena, being conical angle and spiral growth, as given below.

$$
\begin{equation*}
\mathbf{A}=\mathbf{A}_{0} \mathbf{A}_{c} \mathbf{A}_{s} \tag{15}
\end{equation*}
$$

Where $\mathbf{A}_{c}$ and $\mathbf{A}_{s}$ are given by (16).

$$
\mathbf{A}_{c}=\left[\begin{array}{ccc}
\cos \phi & -\sin \phi & 0  \tag{16}\\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{array}\right] \quad \mathbf{A}_{s}=\left[\begin{array}{ccc}
\cos \varpi & 0 & \sin \varpi \\
0 & 1 & 0 \\
-\sin \varpi & 0 & \cos \varpi
\end{array}\right]
$$

$\phi$ is the conical angle and $\varpi$ is the spiral grain angle. Both angles are illustrated in figure 2


Figur 2: conical angle and spiral growth

